

\tilde{Q} – REPRESENTATION OF REAL NUMBERS AND FRACTAL PROBABILITY DISTRIBUTIONS

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ABSTRACT. A \tilde{Q} –representation of real numbers is introduced as a generalization of the s –adic expansion. It is shown that the \tilde{Q} –representation is a convenient tool for the construction and study of fractals and measures with complicated local structure. Distributions of random variables ξ with independent \tilde{Q} –symbols are studied in details. Necessary and sufficient conditions for the corresponding probability measures μ_ξ to be either absolutely continuous or singular (resp. pure continuous, or pure point) are found in terms of the \tilde{Q} –representation. The metric, topological, and fractal properties for the distribution of ξ are investigated. A number of examples are presented.

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1. INTRODUCTION

As well known there exist only three types of pure probability distributions: discrete, absolutely continuous and singular. During a long period mathematicians had a rather low interest in singular probability distributions, which was mainly caused by the two following reasons: the absence of effective analytic tools and the widely spread point of view that such distributions do not have any applications, in particular in physics, and are interesting only for theoretical reasons. The interest in singular probability distributions increased however in 1990's due their deep connections with the theory of fractals. On the other hand, recent investigations show that singularity is generic for many classes of random variables, and absolutely continuous and discrete distributions arise only in exceptional cases (see, e.g. [10, 15]). Possible applications in the spectral theory of self-adjoint operators ([14]) is

an additional reason in the intensive investigation of singular continuous measures. It was proven that Schrödinger type operators with singular continuous spectra are generic for some classes of potentials ([6]). Moreover, by using the fractal analysis of the corresponding spectral singular continuous measures, it is possible to analyze the dynamical properties of the corresponding quantum systems ([9]).

Usually the singular probability distributions are associated with the Cantor-like distributions. Such distributions are supported by nowhere dense sets of zero Lebesgue measures. In the sequel we shall call such distributions the distributions of the *C-type*. But there exist singular probability distributions with other metric-topological properties of their topological support.

A set $E \subseteq R^1$ is said to be of the pure *S-type* if it is a union of an at most countable family of closed intervals $[a_i, b_i]$, $a_i < b_i$. A set $E \subseteq R^1$ is said to be of the pure *C-type* if it is a nowhere dense set of zero Lebesgue measure. A set $E \subseteq R^1$ is said to be of the pure *P-type* if it is a nowhere dense set such that for any $x \in E$ and for any $\varepsilon > 0$ the set $E \cap (x - \varepsilon, x + \varepsilon)$ is of positive Lebesgue measure.

A probability measure μ on R^1 is said to be of the pure S-type if the measure is supported by a set of the pure S-type, i.e., if there exists a set S of the pure S-type such that the topological support (the minimal closed set supporting the measure) $S_\mu = S^{cl}$ and $\mu(S) = 1$, where *cl* stands for the closure.

A probability measure μ on R^1 is said to be of the pure C-type if its topological support S_μ is a nowhere dense set and the measure is supported by a set of the pure C-type, i.e., there exists a set C of C-type such that $S_\mu = C^{cl}$ is a nowhere dense set and $\mu(C) = 1$.

A probability measure μ on R^1 is said to be of the pure P-type if its topological support S_μ is a nowhere dense set and the measure is supported by a set of the pure P-type, i.e., if there exists a set P of the pure P-type such that $S_\mu = P^{cl}$ is a nowhere dense set and $\mu(P) = 1$.

In [10] it has been proved that any *singular continuous* probability measure μ_{sing} on R^1 can be represented in the following form:

$$\mu_{\text{sing}} = \alpha_1 \mu_{\text{sing}}^C + \alpha_2 \mu_{\text{sing}}^S + \alpha_3 \mu_{\text{sing}}^P,$$

where $\alpha_i \geq 0, \alpha_1 + \alpha_2 + \alpha_3 = 1$, μ_{sing}^C , μ_{sing}^S and μ_{sing}^P are singular continuous probability measures of the pure *C*-, *S*- resp. *P*-types. A corresponding decomposition holds for the singular component of the spectral measure and the Hilbert space for any given self-adjoint operator with a simple spectrum (see [2] for details).

In the paper we introduce into consideration the so-called \tilde{Q} -representation of real numbers which is a convenient tool for the construction of a wide class of fractals. This class contains Cantor-like sets as well as everywhere dense noncompact fractals with any desirable Hausdorff-Besicovitch dimension $\alpha_0 \in [0, 1]$. By using the \tilde{Q} -representation we introduce a family of random variables with independent \tilde{Q} -symbols. This family contains all possible the above mentioned types of singular continuous measures, and (as a very particular case) the class of all self-similar measures on $[0, 1]$ satisfying the open set condition (see section 4 for details).

An additional reason for the investigation of the distribution of the random variables with independent \tilde{Q} -symbols is to extend the so-called Jessen-Wintner theorem to the case of sums of random variables which are not independent. In fact this theorem asserts that if a random variable is a sum of the convergent series

of independent discretely distributed random variables, then it has a pure distribution. In general setting, necessary and sufficient conditions for such probability distributions to be singular resp. absolutely continuous are still unknown.

In this paper we completely investigated the structure of the random variables with independent \tilde{Q} -symbols. The topological, metric and fractal properties of the above mentioned probability distributions are also studied.

2. \tilde{Q} -REPRESENTATION OF REAL NUMBERS

We describe the notion of the \tilde{Q} -representation for real numbers $x \in [0, 1]$. Let us consider a $\mathbf{N}_k \times \mathbf{N}$ -matrix $\tilde{Q} = \|q_{ik}\|$, $i \in \mathbf{N}_k$, $k \in \mathbf{N}$, where \mathbf{N} stands for the set of natural numbers and $\mathbf{N}_k = \{0, 1, \dots, N_k\}$, with $0 < N_k \leq \infty$. We suppose that

$$(1) \quad q_{ik} > 0, \quad i \in \mathbf{N}_k, \quad k \in \mathbf{N}.$$

Besides, we assume that for each $k \in \mathbf{N}$:

$$(2) \quad \sum_{i \in \mathbf{N}_k} q_{ik} = 1,$$

and

$$(3) \quad \prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{q_{ik}\} = 0.$$

Given a \tilde{Q} -matrix we consecutively perform decompositions of the segment $[0, 1]$ as follows.

Step 1. We decompose $[0, 1]$ (from the left to the right) into the union of closed intervals $\Delta_{i_1}^{\tilde{Q}}$, $i_1 \in \mathbf{N}_1$ (without common interior points) of the length $|\Delta_{i_1}^{\tilde{Q}}| = q_{i_1 1}$,

$$[0, 1] = \bigcup_{i_1 \in \mathbf{N}_1} \Delta_{i_1}^{\tilde{Q}}.$$

Each interval $\Delta_{i_1}^{\tilde{Q}}$ is called a *1-rank interval*.

Step $k \geq 2$. We decompose (from the left to the right) each closed $(k-1)$ -rank interval $\Delta_{i_1 i_2 \dots i_{k-1}}^{\tilde{Q}}$ into the union of closed k -rank intervals $\Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}}$,

$$\Delta_{i_1 i_2 \dots i_{k-1}}^{\tilde{Q}} = \bigcup_{i_k \in \mathbf{N}_k} \Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}},$$

where their lengths

$$(4) \quad \left| \Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}} \right| = q_{i_1 1} \cdot q_{i_2 2} \cdot \dots \cdot q_{i_k k} = \prod_{s=1}^k q_{i_s s}$$

are related as follows

$$\left| \Delta_{i_1 i_2 \dots i_{k-1} 0}^{\tilde{Q}} \right| : \left| \Delta_{i_1 i_2 \dots i_{k-1} 1}^{\tilde{Q}} \right| : \dots : \left| \Delta_{i_1 i_2 \dots i_{k-1} i_k}^{\tilde{Q}} \right| : \dots = q_{0k} : q_{1k} : \dots : q_{i_k k} : \dots$$

For any sequence of indices $\{i_k\}$, $i_k \in \mathbf{N}_k$, there corresponds the sequence of embedded closed intervals

$$\Delta_{i_1}^{\tilde{Q}} \supset \Delta_{i_1 i_2}^{\tilde{Q}} \supset \dots \supset \Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}} \supset \dots$$

such that $|\Delta_{i_1 \dots i_k}^{\tilde{Q}}| \rightarrow 0$, $k \rightarrow \infty$, due to (3) and (4). Therefore, there exists a unique point $x \in [0, 1]$ belonging to all intervals $\Delta_{i_1}^{\tilde{Q}}$, $\Delta_{i_1 i_2}^{\tilde{Q}}$, ..., $\Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}}$, Conversely, for any point $x \in [0, 1]$ there exists a sequence of embedded intervals $\Delta_{i_1}^{\tilde{Q}} \supset \Delta_{i_1 i_2}^{\tilde{Q}} \supset \dots \supset \Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}} \supset \dots$ containing x , i.e.,

$$(5) \quad x = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}} = \bigcap_{k=1}^{\infty} \Delta_{i_1(x) i_2(x) \dots i_k(x)}^{\tilde{Q}} =: \Delta_{i_1(x) i_2(x) \dots i_k(x) \dots}^{\tilde{Q}}$$

Notation (5) is called the \tilde{Q} -representation of the point $x \in [0, 1]$.

Remark 1. The correspondence $[0, 1] \ni x \Leftrightarrow \{(i_1(x), i_2(x), \dots, i_k(x), \dots)\}$ in (5) is one-to-one, i.e., the \tilde{Q} -representation is unique for every point $x \in [0, 1]$, provided that the \tilde{Q} -matrix contains an infinite number of columns with an infinite number of elements. However in the case, where $N_k < \infty$, $\forall k > k_0$ for some k_0 , there exists a countable set of points $x \in [0, 1]$ having two different \tilde{Q} -representations. Precisely, this is the set of all end-points of intervals $\Delta_{i_1 i_2 \dots i_k}^{\tilde{Q}}$ with $k > k_0$.

One has the formula

$$(6) \quad x = D_1(x) + \sum_{k=2}^{\infty} \left[D_k(x) \prod_{s=1}^{k-1} q_{i_s(x)s} \right] = \sum_{k=1}^{\infty} D_k(x) L_{k-1}(x)$$

$$\text{where } D_k(x) := \begin{cases} 0, & \text{if } i_k(x) = 0, \\ \sum_{i=0}^{i_k(x)-1} q_{ik}, & \text{if } i_k(x) \geq 1 \end{cases} \quad \text{and where we put}$$

$$L_{k-1}(x) := |\Delta_{i_1(x) \dots i_{k-1}(x)}^{\tilde{Q}}| = \prod_{s=1}^{k-1} q_{i_s(x)s}$$

for $k > 1$, and $L_{k-1}(x) = 1$, if $k = 1$.

We note that (6) follows from (5) since the common length of all intervals lying on the left side of a point $x = \Delta_{i_1(x) \dots i_k(x) \dots}^{\tilde{Q}}$ can be calculate as the sum of all 1-rank intervals lying on the left from x (it is the first term $D_1(x)$ in (6)), plus the sum of all 2-rank intervals from $\Delta_{i_1(x)}^{\tilde{Q}}$, lying on the left side from x (the second term $D_2(x) \cdot q_{i_1(x)1}$ in (6)), and so on.

Remark 2. If $q_{ik} = q_i$, $k \in \mathbf{N}$, then the \tilde{Q} -representation coincides with the Q -representation (see [12]); moreover, if $q_{ik} = \frac{1}{s}$ for some natural number $s > 1$, then the \tilde{Q} -representation coincides with the classical s -adic expansion.

3. $\tilde{Q}(\mathbf{V})$ -REPRESENTATION FOR FRACTALS

The \tilde{Q} -representation allows to construct in a convenient way a wide class of fractals on R^1 and other mathematical objects with fractal properties. Firstly we consider compact fractals from R^1 . Let $\mathbf{V} := \{\mathbf{V}_k\}_{k=1}^{\infty}$, $\mathbf{V}_k \subseteq \mathbf{N}_k$, and let us consider the set

$$(7) \quad \Gamma_{\tilde{Q}(\mathbf{V})} \equiv \Gamma := \left\{ x \in [0, 1] : x = \Delta_{i_1 i_2 \dots i_k \dots}^{\tilde{Q}}, i_k \in \mathbf{V}_k \right\},$$

i.e., Γ consists of points, which can be \tilde{Q} -represented by using only symbols i_k from the set \mathbf{V}_k on each k -th position of their \tilde{Q} -representation.

If $\mathbf{V}_k \neq \mathbf{N}_k$ at least for one $k < k_0$, and $\mathbf{V}_k = \mathbf{N}_k$ for all $k \geq k_0$ with some fixed $k_0 > 1$, then Γ is a union of closed intervals. In this case one can get Γ removing from $[0, 1]$ all open intervals $\Delta_{i_1 \dots i_k}^{\tilde{Q}}$, $k < k_0$ with $i_k \notin \mathbf{V}_k$ (where a point over Δ means that an interval is open). If the condition $\mathbf{V}_k \neq \mathbf{N}_k$ holds for infinitely many values of k , then obviously Γ is a nowhere dense set.

We shall study the metric properties of the sets $\Gamma_{\tilde{Q}(\mathbf{V})}$. Let $S_k(\mathbf{V})$ denote the sum of all elements q_{ik} such that $i_k \in \mathbf{V}_k$, i.e.,

$$S_k(\mathbf{V}) := \sum_{i \in \mathbf{V}_k} q_{ik}.$$

We note that $0 < S_k(\mathbf{V}) \leq 1$ due to (1), (2).

Lemma 1. *Lebesgue measure $\lambda(\Gamma)$ of the set Γ is equal to*

$$(8) \quad \lambda(\Gamma) = \prod_{k=1}^{\infty} S_k(\mathbf{V}).$$

Proof. Let $\Gamma_n := \bigcup_{i_k \in \mathbf{V}_k} \Delta_{i_1 \dots i_n}$. It is easy to see that $\Gamma_n \subseteq \Gamma_{n-1}$ and $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$.

From the definition of the sets Γ_n and from (4), it follows that $\lambda(\Gamma_n) = \prod_{k=1}^n S_k(\mathbf{V})$,

and, therefore, $\lambda(\Gamma) = \lim_{n \rightarrow \infty} \lambda(\Gamma_n) = \prod_{k=1}^{\infty} S_k(\mathbf{V})$. □

Lemma 2. *Let $W_k(\mathbf{V}) = 1 - S_k(\mathbf{V}) \geq 0$. The set Γ is of zero Lebesgue measure if and only if*

$$(9) \quad \sum_{k=1}^{\infty} W_k(\mathbf{V}) = \infty.$$

Proof. This assertion is a direct consequence of the previous lemma and the well known relation between infinite products and infinite series. Namely, for a sequence $0 \leq a_k < 1$, the product $\prod_{k=1}^{\infty} (1 - a_k) = 0$ if and only if the sum $\sum_{k=1}^{\infty} a_k = \infty$. In our case $a_k = 1 - S_k(\mathbf{V})$. □

The above mentioned procedure allows to construct nowhere dense compact fractal sets E with desirable Hausdorff-Besicovitch dimension (including the anomalously fractal case ($\alpha_0(E) = 0$) and the superfractal case ($\alpha_0(E) = 1$)) in a very compact way.

Theorem 1. *Let $\mathbf{N}_k = N_{s-1}^0 := \{0, 1, \dots, s-1\}$ $k \in N$, let $\mathbf{V}_0 = \{v_1, v_2, \dots, v_m\} \subset N_{s-1}^0$ and let the matrix \tilde{Q} have the following asymptotic property:*

$$\lim_{k \rightarrow \infty} q_{ik} = q_i, \quad i \in N_{s-1}^0.$$

Then:

1) *the Hausdorff-Besicovitch dimension of the set $\Gamma_{\tilde{Q}(\mathbf{V}_0)}$ is the root of the following equation*

$$(10) \quad \sum_{i \in \mathbf{V}_0} q_i^x = 1, \quad ;$$

2) if

$$M[\tilde{Q}, (\nu_0, \dots, \nu_{s-1})] = \left\{ x : \Delta_{\alpha_1(x) \dots \alpha_k(x) \dots}^{\tilde{Q}}, \lim_{k \rightarrow \infty} \frac{N_i(x, k)}{k} = \nu_i, i \in N_{s-1}^0 \right\},$$

where $N_i(x, k)$ is the number of symbols "i" among the first k symbols of the \tilde{Q} -representation of x , then

$$(11) \quad \alpha_0(M[\tilde{Q}, (\nu_0, \dots, \nu_{s-1})]) = \frac{\sum_{i=0}^{s-1} \nu_i \ln \nu_i}{\sum_{i=0}^{s-1} \nu_i \ln q_i}.$$

Proof. Firstly we consider the particular case where the matrix \tilde{Q} has exactly s rows and all its columns are the same: $q_{ik} = q_i$. In such a simple case the \tilde{Q} -representation reduces to the Q -representation studied in [10]. It is easy to prove (see, e.g., [12]), that to calculate the Hausdorff-Besicovitch dimension of any subset $E \subset [0, 1]$ it is sufficient to consider a class of cylinder sets of different ranks generated by Q -partitions of the unit interval. The Billingsley theorem (see, e.g., [4], p.141) admits a generalization to the class of Q -cylinders, and, from this theorem it follows that in the case of usual Q -representation, the Hausdorff-Besicovitch dimension of the set $M[Q, (\nu_0, \dots, \nu_{s-1})]$ is equal to the right-side expression in (11).

In the Q -case the set $\Gamma_{Q(\mathbf{V}_0)}$ is a self-similar set satisfying the open set condition. Therefore, the Hausdorff-Besicovitch dimension of this set is the root of equation (10).

Now let us consider a general case of theorem 1. To this end we introduce into consideration the following transformation f of $[0, 1]$:

$$f(x) = f(\Delta_{\alpha_1(x) \dots \alpha_k(x) \dots}^Q) = \Delta_{\alpha_1(x) \dots \alpha_k(x) \dots}^{\tilde{Q}}.$$

It is not hard to prove, that such a transformation belongs to the DP-class (see, e.g., [3]), i.e., f preserves the Hausdorff-Besicovitch dimension of any subset of $[0, 1]$. Since $f(\Gamma_{Q(\mathbf{V}_0)}) = \Gamma_{\tilde{Q}(\mathbf{V}_0)}$ and $f(M[Q, (\nu_0, \dots, \nu_{s-1})]) = M[\tilde{Q}, (\nu_0, \dots, \nu_{s-1})]$, we get the desired formulas under general assumptions of theorem 1. \square

Example 1. If $\mathbf{N}_k = \{0, 1, 2\}$, $\mathbf{V}_k = \{0, 2\}$, $q_{1k} \rightarrow 0$, but $\sum_{k=1}^{\infty} q_{1k} = \infty$ with $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, then Γ is a nowhere dense set of zero Lebesgue measure. From Theorem 1 it follows that the Hausdorff dimension of this set is equal 1. In the terminology of [10] a set of this kind is called a superfractal set.

Example 2. If $\mathbf{N}_k = \{0, 1, 2\}$, $\mathbf{V}_k = \{0, 2\}$, $q_{1k} \rightarrow 1$ (but $\prod_{k=1}^{\infty} q_{1k} = 0$), and $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, then Γ is a nowhere dense set of zero Lebesgue measure and of zero Hausdorff dimension, i.e., Γ is an anomalously fractal set (see [10]).

4. RANDOM VARIABLES WITH INDEPENDENT \tilde{Q} -SYMBOLS

Let $\{\xi_k, k \in \mathbf{N}\}$ be a sequence of independent random variables with the following distributions

$$P(\xi_k = i) := p_{ik} \geq 0, \text{ with } \sum_{i \in \mathbf{N}_k} p_{ik} = 1, k \in \mathbf{N}.$$

By using ξ_k and the \tilde{Q} -representation we construct a random variable ξ as follows:

$$(12) \quad \xi := \Delta_{\xi_1 \xi_2 \dots \xi_k \dots}^{\tilde{Q}}.$$

The distribution of ξ is completely determined by two matrices: \tilde{Q} and $\tilde{P} = ||p_{ik}||$, where some elements of the matrix \tilde{P} possibly are equal to zero. Of course, all sets \mathbf{N}_k are the same as those in the \tilde{Q} -matrix. Let μ_ξ be the measure corresponding the distribution of the random variable ξ with independent \tilde{Q} -symbols.

If $q_{ik} = q_i$ and $p_{ik} = p_i \forall j \in N, i \in N_{s-1}^0$ (i.e., ξ is a random variable with independent identically distributed Q -digits), then the measure μ_ξ is the self-similar measure associated with the list $(S_1, \dots, S_{s-1}, p_1, \dots, p_{s-1})$, where S_i is the similarity with the ratio q_i ($\sum_{i=0}^{s-1} q_i = 1$), and the list (S_1, \dots, S_{s-1}) satisfies the open set condition. More precisely, μ_ξ is the unique Borel probability measure on $[0, 1]$ such that

$$\mu_\xi = \sum_{i=0}^{s-1} p_i \cdot \mu_\xi \circ S_i^{-1},$$

(see, e.g., [7] for details). In the so-called " Q^* -case" we construct the measure μ_ξ in a similar way but with the possibility of changing of the ratios and probabilities from the list $(S_1, \dots, S_{s-1}, p_1, \dots, p_{s-1})$ at each stage of the construction. In our general " \tilde{Q} -case" we may additionally choose the number of contracting similarities (including a countable number) at each stage of the construction.

From (6) it follows that ξ can be represented as a sum of the convergent series of discretely distributed random variables which are not independent. Nevertheless the distribution of ξ is of pure type.

Theorem 2. *The measure μ_ξ is of pure type, i.e., it is either purely absolutely continuous, resp., purely point, resp., purely singular continuous. Precisely,*

1) μ_ξ is purely absolutely continuous if and only if

$$(13) \quad \rho := \prod_{k=1}^{\infty} \left\{ \sum_{i \in \mathbf{N}_k} \sqrt{p_{ik} \cdot q_{ik}} \right\} > 0;$$

2) μ_ξ is purely point if and only if

$$(14) \quad P_{max} := \prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{p_{ik}\} > 0;$$

3) μ_ξ is purely singular continuous if and only if

$$(15) \quad \rho = 0 = P_{max}.$$

Proof. Let $\Omega_k = \mathbf{N}_k$, $\mathcal{A}_k = 2^{\Omega_k}$. We define measures μ_k and ν_k in the following way:

$$\mu_k(i) = p_{ik}; \quad \nu_k(i) = q_{ik}, \quad i \in \Omega_k.$$

Let

$$(\Omega, \mathcal{A}, \mu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \mu_k), \quad (\Omega, \mathcal{A}, \nu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \nu_k)$$

be the infinite products of probability spaces, and let us consider the measurable mapping $f : \Omega \rightarrow [0; 1]$ defined as follows:

$$\forall \omega = (\omega_1, \omega_2, \dots, \omega_k, \dots) \in \Omega, \quad f(\omega) = x = \Delta_{i_1(x)i_2(x)\dots i_k(x)\dots}$$

with $\omega_k = i_k(x)$ $k \in \mathbf{N}$.

We define the measures μ^* and ν^* as the image measure of μ resp. ν under f :

$$\mu^*(B) := \mu(f^{-1}(B)); \quad \nu^*(B) = \nu(f^{-1}(B)), B \in \mathcal{B}.$$

It is easy to see that ν^* coincides with Lebesgue measure λ on $[0, 1]$, and $\mu^* \equiv \mu_{\xi}$. In general, the mapping f is not bijective, but there exists a countable set Ω_0 such that $\nu(\Omega_0) = 0, \mu(\Omega_0) = 0$ and the mapping $f : \Omega \setminus \Omega_0 \rightarrow [0, 1]$ is bijective.

Therefore, the measure μ_{ξ} is absolutely continuous (singular) with respect to Lebesgue measure if and only if the measure μ is absolutely continuous (singular) with respect to the measure ν . Since, $q_{ik} > 0$, we conclude that $\mu_k \ll \nu_k, \forall k \in \mathbf{N}$. By using Kakutani's theorem [8], we have

$$(16) \quad \mu_{\xi} \ll \lambda \Leftrightarrow \prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k > 0 \Leftrightarrow \prod_{k=1}^{\infty} \left(\sum_{i \in \mathbf{N}_k} \sqrt{p_{ik}q_{ik}} \right) > 0,$$

$$(17) \quad \mu_{\xi} \perp \lambda \Leftrightarrow \prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k = 0 \Leftrightarrow \prod_{k=1}^{\infty} \left(\sum_{i \in \mathbf{N}_k} \sqrt{p_{ik}q_{ik}} \right) = 0.$$

Of course, a singularly distributed random variable ξ can also be distributed discretely. For any point $x \in [0, 1]$ the set $f^{-1}(x)$ consists of at most two points from Ω . Therefore, the measure μ_{ξ} is an atomic measure if and only if the measure μ is atomic.

If $\prod_{k=1}^{\infty} \max_i p_{ik} = 0$, then

$$\mu(\omega) = \prod_{k=1}^{\infty} p_{\omega_k k} \leq \prod_{k=1}^{\infty} \max_i p_{ik} = 0 \quad \text{for any } \omega \in \Omega,$$

and μ is continuous.

If $\prod_{k=1}^{\infty} \max_i p_{ik} > 0$, then we consider the subset $A_+ = \{\omega : \mu(\omega) > 0\}$. The set A_+ contains the point $\omega^* = (\omega_1^*, \omega_2^*, \dots, \omega_k^*, \dots)$ such that $p_{\omega_k^* k} = \max_i p_{ik}$. It is easy to see that for all $\omega \in A_+$ the condition $p_{\omega_k k} \neq \max_i p_{ik}$ holds only for a finite amount of values k . This means that A_+ is a countable set and the event " $\omega \in A_+$ " does not depend on any finite coordinates of ω . Therefore, by using Kolmogorov's "0 and 1" theorem, we conclude that $\mu(A_+) = 0$ or $\mu(A_+) = 1$. Since $\mu(A_+) \geq \mu(\omega^*) > 0$, we have $\mu(A_+) = 1$, which proves the discreteness of the measure μ . \square

Remark 3. If there exists a positive number q^+ such that $q_{ik} \geq q^+, \forall k \in \mathbf{N}, \forall i \in \mathbf{N}_k$, then condition (16) is equivalent to the convergence of the following series:

$$(18) \quad \sum_{k=1}^{\infty} \left\{ \sum_{i \in \mathbf{N}_k} \left(1 - \frac{p_{ik}}{q_{ik}}\right)^2 \right\} < \infty.$$

If $\lim_{k \rightarrow \infty} q_{ik} = 0$, then, generally speaking, conditions (16) and (18) are not equivalent. For example, let us consider the matrices \tilde{Q} and \tilde{P} as follows: $\mathbf{N}_k = \{0, 1, 2\}$, $q_{1k} = \frac{1}{2k}$, $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, $p_{1k} = 0$, $p_{0k} = p_{2k} = \frac{1}{2}$. In this case condition (16) holds, but (18) does not hold.

5. METRIC-TOPOLOGICAL CLASSIFICATION AND FRACTAL PROPERTIES OF THE DISTRIBUTIONS OF THE RANDOM VARIABLES WITH INDEPENDENT \tilde{Q} -SYMBOLS

For any probability distribution there exist sets which essentially characterize the properties of the distribution. We would like to stress the role of the following sets.

a) Topological support $S_\psi = \{x : F(x + \varepsilon) - F(x - \varepsilon) > \varepsilon, \forall \varepsilon > 0\}$. S_ψ is the smallest closed support of the distribution of ψ .

b) Essential support $N_\psi^\infty = \left\{x : \lim_{\varepsilon \rightarrow 0} \frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon} = +\infty\right\}$.

If the topological support of a distribution is a fractal, then the corresponding distribution is said to be externally fractal. The probability distribution of a random variable ψ is said to be internally fractal if the essential support of the distribution is a fractal set.

First of all we shall analyze the metric and topological properties of the topological support of the random variable with independent \tilde{Q} -symbol. In [2, 10] it was proven that arbitrary singular continuous probability measures can be decomposed into linear combinations of singular probability measures of S-, C- and P-types (see the Introduction for the definitions).

We shall prove now that the above considered probability measures μ_ξ are of the pure above mentioned metric-topological types. Moreover we give necessary and sufficient conditions for a probability measure to belong to each of these types.

Theorem 3. *The distribution of the random variable ξ with independent \tilde{Q} -symbols has pure metric-topological type. Namely, the measure μ_ξ is one of the following three types:*

1) *it is of the pure S-type if and only if the matrix \tilde{P} contains only a finite number of columns containing zero elements;*

2) *it is of the pure C-type if and only if the matrix \tilde{P} contains infinitely many columns having some elements $p_{ik} = 0$, and besides*

$$(19) \quad \sum_{k=1}^{\infty} \left(\sum_{i: p_{ik}=0} q_{ik} \right) = \infty.$$

3) *it is of the pure P-type if and only if the matrix \tilde{P} contains infinitely many columns having zero elements and besides*

$$(20) \quad \sum_{k=1}^{\infty} \left(\sum_{i: p_{ik}=0} q_{ik} \right) < \infty;$$

Proof. Let us consider the set $\Gamma \equiv \Gamma_{\tilde{Q}(\mathbf{V})}$ (see Sect. 3) with $\mathbf{V} = \{\mathbf{V}_k\}_{k=1}^\infty$ defined by the \tilde{P} -matrix as follows: $\mathbf{V}_k = \{i \in \mathbf{N}_k : p_{ik} \neq 0\}$. It is easy to see that the topological support of the measure μ_ξ coincides with a set Γ or its closure, i.e.,

$$(21) \quad S_\xi = \Gamma_{\tilde{Q}(\mathbf{V})}^{cl}.$$

Therefore to examine the metric-topological structure of the set S_ξ we may apply the results of section 3. So, if the matrix \tilde{P} contains only finite number of zero elements, then $\mathbf{V}_k = \mathbf{N}_k$, $k > k_0$ for some $k_0 > 0$. In such a case, Γ is a union of at most countable family of closed intervals. Hence (21) implies that the measure μ_ξ is of the pure S-type.

In the opposite case where the matrix \tilde{P} contains an infinite number of columns where some elements $p_{ik} = 0$, then obviously Γ is a nowhere dense set (see Sect. 3). The Lebesgue measure of the set Γ by Lemma 1 is equal to

$$\lambda(\Gamma) = \prod_{k=1}^{\infty} S_k(\mathbf{V}) = \prod_{k=1}^{\infty} \left(\sum_{i \in \mathbf{V}_k} q_{ik} \right) = \prod_{k=1}^{\infty} \left(1 - \sum_{i: p_{ik}=0} q_{ik} \right).$$

Then, by Lemma 2, either $\lambda(\Gamma) = 0$, provided that condition (19) fulfilled, or $\lambda(\Gamma) > 0$, if condition (20) holds. Thus the measure μ_ξ either is of the C-type, or it is of the P-type.

Since the conditions 1), 2) and 3) of this theorem are mutually exclusive and one of them always holds, we conclude that the distribution of the random variable ξ with independent \tilde{Q} -symbols always has a pure metric-topological type. \square

By using the latter theorems we can construct measures of **8** kinds: pure point as well as pure singular continuous of any S-, C-, or P-types, and pure absolutely continuous but only of the S- and P-types.

We illustrate this statement by examples.

Example 3.

Let $\mathbf{N}_k = \{0, 1, 2\}$ and let the \tilde{Q} -matrix be given by $q_{0k} = q_{1k} = q_{2k} = \frac{1}{3}$, $k \in N$.

S_{pp} : If $p_{0k} = \frac{1-p_{1k}}{2}$, $p_{1k} = 1 - \frac{1}{2^k}$, $p_{2k} = \frac{1-p_{1k}}{2}$, then μ_ξ is a discrete measure of the pure S-type. In this case $S_\xi = [0, 1]$ and N_ξ^∞ is a countable set which is dense on $[0, 1]$.

S_{sc} : If $p_{0k} = \frac{1}{4}$, $p_{1k} = \frac{1}{2}$, $p_{2k} = \frac{1}{4}$, then μ_ξ is a singular continuous measure of pure S-type. In this case again $S_\xi = [0, 1]$ but N_ξ^∞ is now a fractal set which is also dense on $[0, 1]$.

S_{ac} : If $p_{0k} = p_{1k} = p_{2k} = \frac{1}{3}$, then μ_ξ coincides with Lebesgue measure on $[0, 1]$.

Example 4.

Let again $\mathbf{N}_k = \{0, 1, 2\}$ and let the \tilde{Q} -matrix be given by $q_{0k} = q_{1k} = q_{2k} = \frac{1}{3}$, $k \in N$. Then

C_{pp} : If $p_{0k} = 1 - \frac{1}{2^k}$, $p_{1k} = 0$, $p_{2k} = \frac{1}{2^k}$, then μ_ξ is a pure point measure of the pure C-type. In this case $S_\xi \equiv C_0$ coincides with the classical Cantor set C_0 and its essential support is a countable set which is dense on C_0 .

C_{sc} : If $p_{0k} = \frac{1}{2}$, $p_{1k} = 0$, $p_{2k} = \frac{1}{2}$, then μ_ξ is the classical Cantor measure.

Example 5.

Let as above $\mathbf{N}_k = \{0, 1, 2\}$ and let the \tilde{Q} -matrix be given by $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, $q_{1k} = \frac{1}{2^k}$, $k \in N$. Then

P_{pp} : If $p_{0k} = 1 - \frac{1}{2^k}$, $p_{1k} = 0$, $p_{2k} = \frac{1}{2^k}$, then μ_ξ is a pure point measure of the pure P-type.

P_{sc} : If $p_{0k} = \frac{1}{4}$, $p_{1k} = 0$, $p_{2k} = \frac{3}{4}$, then μ_ξ is a singular continuous measure of the pure P-type.

P_{ac} : If $p_{0k} = p_{2k} = \frac{1}{2}$, $p_{1k} = 0$, then μ_ξ is of the pure P-type measure which is absolutely continuous w.r.t. Lebesgue measure.

We would like to stress that the essential support is more suitable to describe the properties of distributions with complicated local structure. As we saw above, a discrete probability distribution may be of C -, P - resp. S -type, and the topological support of discrete distribution can be of any Hausdorff-Besicovitch dimension $\alpha_0 \in [0, 1]$. But the essential support of a discrete distribution is always at most countable set.

The essential support is especially suitable for singular distributions because of the following fact: a random variable ψ is singularly distributed iff $P_\psi(N_\psi^\infty) = 1$.

For an absolutely continuous distribution the topological support is always of positive Lebesgue measure. But the essential support may be of very complicated local structure. In [3] we constructed an example of an absolutely continuous distribution function such that the essential support is an everywhere dense superfractal set ($\alpha_0(N_\xi^\infty) = 1$). Therefore, the condition $\alpha_0(N_\xi^\infty) > 0$ does not imply the singularity of the distribution.

The following notion is very important for describing the fractal properties of probability distributions. Let A_ξ be the set of all possible supports of the distribution of the r.v. ξ , i.e.,

$$A_\xi = \{E : E \in \mathcal{B}, P_\xi(E) = 1\}.$$

The number $\alpha_0(\xi) = \inf_{E \in A_\xi} \{\alpha_0(E)\}$ is said to be the Hausdorff-Besicovitch dimension of the distribution of the r.v. ξ .

It is obvious that $\alpha_0(\xi) = 0$ for any discrete distribution; on the other hand, $\alpha_0(\xi) = 1$ for any absolutely continuous distribution. $\alpha_0(\xi)$ can be an arbitrary number from $[0, 1]$ for a singular continuous distribution.

The determination of the Hausdorff-Besicovitch dimension of the distribution of a random variable ψ is essentially more difficult problem than the calculation of the fractal dimension of the corresponding topological support, and in general setting the problem of the determination of the Hausdorff-Besicovitch dimension of the distribution of the random variable ξ with independent \tilde{Q} -symbols is still open. The next (and more complicated) stage in the fractal analysis of a singularly distributed random variable ψ is the finding the support E_ψ of ψ such that $\alpha_0(E_\psi) = \alpha_0(\psi)$.

Theorem 4. Let $p_{ik} = p_i$, $q_{ik} = q_i$, $k \in N$, $i \in N_{s-1}^0$, and let

$$M[\tilde{Q}, (p_0, \dots, p_{s-1})] = \left\{ x : \lim_{k \rightarrow \infty} \frac{N_i(x, k)}{k} = p_i, i \in N_{s-1}^0 \right\},$$

where $N_i(x, k)$ is the number of symbols "i" among the first k symbols of the \tilde{Q} -representation of x .

Then $E_\xi = M[\tilde{Q}, (p_0, \dots, p_{s-1})]$ and

$$\alpha_0(\xi) = \frac{\sum_{i=0}^{s-1} p_i \ln p_i}{\sum_{i=0}^{s-1} p_i \ln q_i}.$$

Proof. For the simple Q -representation the problem of the determination of the Hausdorff-Besicovitch dimension of the distribution of the random variable ξ was

solved in [13]. In particular, we have $\alpha_0(\xi) = \frac{\sum_{i=0}^{s-1} p_i \ln p_i}{\sum_{i=0}^{s-1} p_i \ln q_i}$. From Theorem 1 it fol-

lows that the Hausdorff-Besicovitch dimension of the set $M[\tilde{Q}, (p_0, \dots, p_{s-1})]$, which consists of the points whose \tilde{Q} -representation contains the digit "i" with the as-

ymptotic frequency p_i , is equal to $\frac{\sum_{i=0}^{s-1} p_i \ln p_i}{\sum_{i=0}^{s-1} p_i \ln q_i}$. So, the set $M[\tilde{Q}, (p_0, \dots, p_{s-1})]$ can

be considered as the "dimensionally minimal" support of the distribution of the random variable with independent identically distributed Q -symbols \square

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